THE FUNDAMENTAL MIXED PROBLEM OF THE AXISYMMETRIC THEORY OF ELASTICITY

(OSNOVNAIA SMESHANNAIA ZADACHA OSESIMMETRICHNOI TEORII UPRUGOSTI)

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IU. I. SOLOV'EV
(Novosibirsk)

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Analytic and p-analytic functions of a complex variable have been used in the solution of axisymmetric problems in the theory of elasticity (see, for example, [1 to 5]). In [6 and 7], the same results were accomplished by using generalized analytic functions which do not differ essentially from the functions introduced in [8].

In this manner, Fredholm integral equations were obtained for the first and second fundamental problems for simply as well as multiply connected bodies of revolution.

The method given below deals with the fundamental mixed problem in which the applied forces are specified on one part of the boundary while the displacements are given on the other part. The singular integral equation which is obtained is analogous to the corresponding equation in the plane theory of elasticity [9 and 10]. This equation is then investigated, and the existence of a solution is proved.

1. Let D be a symmetric plane region representing the cross-section of a body of revolution, and let L be the boundary of this region consisting of simple, closed curves with no common points. Introduce a zor coordinate system in the plane of the axial cross-section, the z-axis coinciding with the axis of symmetry. The parts of D lying to the right and to the left of the z-axis (Fig. 1) will be designated by D' and D'', respectively. The designations L' and L'' are assigned in a similar manner. The interior contours L_j' and L_j'' $(j = 1, 2, ..., m)$ do not intersect the axis of symmetry. The remaining interior contours L_j $(j = m + 1, ..., n)$ will be numbered from the bottom upwards in the order in which they intersect z. The outside contour L_{n+1} contains within it all the remaining ones. Assume that the curvature of every contour satisfies the condition H (1). The positive direction on L is chosen so that the region D is on the left. The initial points on the L_j' curves $(j = m + 1, ..., n + 1)$ will be designated by x_j and the final points by x_j' (Fig. 1).

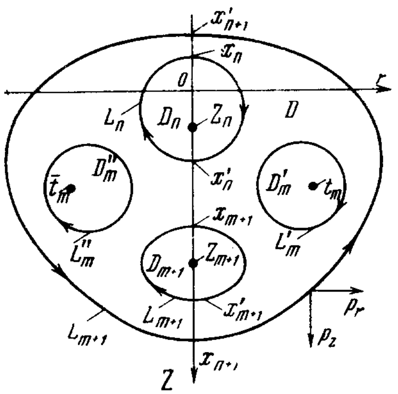

FIG. 1

All considerations presented herein are also applicable (with appropriate modifications) to toroidal bodies ($n = m$ and the contour L_{n+1} decomposes into two closed curves L'_{n+1} and L''_{n+1} having no common points) as well as infinite regions with axisymmetric holes (the contour L_{n+1} is then absent).

As was shown in [6], the general solution of the axisymmetric problem may be written in the form

$$2G(w + iu) = \varkappa'\Phi(t, \bar{t}) - t\overline{\Phi'(t, \bar{t})} - \overline{\Psi(t, \bar{t})} \tag{1.1}$$

Here, w and u are the axial and radial displacements of a point in the elastic body, G is the shear modulus, $\varkappa' = 3.5-4\nu$; ν is Poisson's ratio, while $\Phi(t, \bar{t})$ and $\Psi(t, \bar{t})$ are generalized analytic functions of the arguments $t = z + ir$ and $\bar{t} = z - ir$, and satisfy differential equations of the form

$$2\frac{\partial\Phi}{\partial\bar{t}} - \frac{1}{t-\bar{t}}(\Phi - \overline{\Phi}) = 0 \qquad \left(2\frac{\partial}{\partial\bar{t}} = \frac{\partial}{\partial z} + i\frac{\partial}{\partial r}\right) \tag{1.2}$$

as well as the condition

$$\Phi(t, \bar{t}) = \overline{\Phi(\bar{t}, t)} \tag{1.3}$$

The symbol $\Phi'(t, \bar{t})$ denotes the derivative in the sense of L. Bers with respect to the generating pair $(1, i/r)$, and is numerically equal to $\partial\Phi/\partial z$.

In cylindrical coordinates, the stress components corresponding to the displacements in (1.1) are:

$$\sigma_z + \sigma_r + \sigma_\theta = 2(1+\nu)(\Phi' + \overline{\Phi'}), \quad \sigma_z + \sigma_r = 2(\Phi' + \overline{\Phi'}) - 2G(u/r)$$
$$\sigma_z + i\tau_{zr} = 1{,}5\Phi' + \overline{\Phi'} - t\overline{\Phi''} - \overline{\Psi'} \tag{1.4}$$

Let cd be an arbitrary, smooth curve in D', and let p_z and p_r be the applied loads on the surface of revolution generated by revolving the above curve about the z-axis. Then

$$-R + \frac{i}{r}Z = 0.5\Phi(t, \bar{t}) + t\overline{\Phi'(t, \bar{t})} + \overline{\Psi(t, \bar{t})} - C - \frac{2(1-\nu)}{t-\bar{t}}C' -$$
$$-2(1-\nu)\int_c^t\left[2i\,\mathrm{Im}\,\Phi(t_1, \bar{t}_1) - \frac{1}{t_1-\bar{t}_1}C'\right]\frac{dt_1 + d\bar{t}_1}{t_1 - \bar{t}_1} \tag{1.5}$$

$$Z(s) = \int_0^s p_z(s_1)\,r(s_1)\,ds_1, \qquad R(s) = \int_0^s\left[p_r(s_1) + \frac{1}{r^2(s_1)}Z(s_1)\frac{dz(s_1)}{ds_1}\right]ds_1 \tag{1.6}$$

Here, s is the running coordinate of the point t on the curve, measured from c, and C and C' are real constants.

In order to examine the conditions for single-valuedness and continuity of the stresses and displacements, we use the following representations:

$$\Phi(t, \bar{t}) = \Phi_*(t, \bar{t}) + \sum_{j=1}^{n} A_j\Theta(t, \bar{t}; t_j, \bar{t}_j) + \sum_{j=1}^{m} B_j\Xi(t, \bar{t}; t_j, \bar{t}_j) + \frac{1}{t-\bar{t}}A$$
$$\Psi(t, \bar{t}) = \Psi_*(t, \bar{t}) - \varkappa'\sum_{j=1}^{n} A_j\Theta(t, \bar{t}; t_j, \bar{t}_j) + \varkappa'\sum_{j=1}^{m} B_j\Xi(t, \bar{t}; t_j, \bar{t}_j) - \frac{\varkappa'}{t-\bar{t}}A \tag{1.7}$$

Here A, A_j and B_j are real constants, t_j are arbitrary fixed points inside the respective contours L_j' $(j = 1, 2, \ldots m)$ and L_j $(j = m + 1, \ldots n)$ and Φ_* and Ψ_* are generalized analytic functions, regular in D (i.e. the functions and their derivatives are single-valued and continuous). The functions Ξ and Θ are logarithmic, satisfying Equation (1.2) and condition (1.3); in traversing a contour L_j' in a counterclockwise direction, $\Xi(t, \bar{t}; t_j, \bar{t}_j)$ increases by 2π while $\Theta(t, \bar{t}; t_j, \bar{t}_j)$ increases by $2\pi/(t - \bar{t})$. For Im $t \geqslant 0$ and Im $t_j \geqslant 0$, these functions are given by

$$\Xi(t, \bar{t}; t_j, \bar{t}_j) = -2H(\delta, q) + 2\frac{z - z_j}{|t - \bar{t}_j|}(1 + q)\boldsymbol{K}(q) + \frac{2i|t - \bar{t}_j|}{r(1 + q)}[\boldsymbol{K}(q) - \boldsymbol{E}(q)]$$

$$\Theta(t, \bar{t}; t_j, \bar{t}_j) = \frac{1 + q}{|t - \bar{t}_j|}\boldsymbol{K}(q) + \frac{i}{r}\left[H(\delta, q) - \frac{\pi}{2}\right] \tag{1.8}$$

$$H(\delta, q) = \boldsymbol{E}(q)F(\delta, q') - \boldsymbol{K}(q)F(\delta, q') + \boldsymbol{K}(q)E(\delta, q')$$

$$q = \frac{|t - \bar{t}_j| - |t - t_j|}{|t - \bar{t}_j| + |t - t_j|}, \qquad q' = \sqrt{1 - q^2}, \qquad \delta = \cos^{-1}\left[\frac{q}{q'}\left(\frac{r}{qr_j} - 1\right)^{1/2}\right] \tag{1.9}$$

Here $F(\delta, q')$ and $E(\delta, q')$ are incomplete elliptic integrals of the first and second kind, respectively, with modulus q', and $K(q)$ and $E(q)$ are complete elliptic integrals with modulus q. We confine ourselves to those branches of the functions Ξ and Θ for which $\delta = \pi/2$ if $r = 0$ and $z \to \infty$, while $\delta = -\pi/2$ if $r = 0$ and $z \to -\infty$. The branch cuts connect the points t_j and $\bar{t}_j$ for all $j = 1$ to m, intersecting the z-axis at the same point, which lies in D, above the last contour L_n. When $j \geqslant m + 1$, we set $t = \bar{t}_j = z_j$ and place the branch cut on the z-axis, along $z \leqslant z_j$.

2. Divide L' into $n_1 + 1$ segments l_k' $(0 \leqslant k \leqslant n_1,\ n_1 \geqslant n)$, whose initial points are denoted by c_k. The points c_k and $\bar{c}_k$ which are not on the axis of symmetry will be called nodes; the points where L intersects the z-axis are not included among the nodes. The number of nodes is $2p$ $(p \leqslant n_1)$. Let Λ_1 be the set of curve segments l_k' on which the external loads p_z and p_r are specified, and let Λ_2 be the set of curve segments l_k' on which the displacements w and u are specified. On a given contour L_j segments belonging to Λ_1 alternate with segments belonging to Λ_2. For the present, we will assume that none of the contours L_j' belongs completely to Λ_1. Substituting (1.7) into (1.5) and (1.1), and letting t approach the boundary point $\tau_0 \in l_k'$ $(0 \leqslant k \leqslant n_1)$, we obtain

$$2b(\tau_0)\Phi_*(\tau_0, \bar{\tau}_0) - \varkappa'\Phi_*(\tau_0, \bar{\tau}_0) + \overline{\tau_0\Phi_*'(\tau_0, \bar{\tau}_0)} + \overline{\Psi_*(\tau_0, \bar{\tau}_0)} + \sum_{j=1}^{n} A_j S_j(\tau_0) +$$

$$+ \sum_{j=1}^{m} B_j T_j(\tau_0) - b(\tau_0)\int_{c_k}^{\tau_0} 2i\,\mathrm{Im}\,\Phi_*(\tau, \bar{\tau})\frac{d\tau + d\bar{\tau}}{\tau - \bar{\tau}} = -f(\tau_0) + C(\tau_0) \tag{2.1}$$

Here

$$S_j(\tau_0) = 2b(\tau_0)\Theta_j - \varkappa'(\Theta_j + \bar{\Theta}_j) + \tau_0\bar{\Theta}_j' - \frac{1}{\tau_0 - \bar{\tau}_0}\alpha_j(\tau_0)b(\tau_0) -$$

$$- b(\tau_0)\int_{c_k}^{\tau_0}\left[2i\,\mathrm{Im}\,\Theta(\tau, \bar{\tau}; t_j, \bar{t}_j) - \frac{1}{\tau - \bar{\tau}}\alpha_j(\tau_0)\right]\frac{d\tau + d\bar{\tau}}{\tau - \bar{\tau}} \tag{2.2}$$

$$T_j(\tau_0) = 2b(\tau_0)\Xi_j - \varkappa'(\Xi_j - \bar{\Xi}_j) + \tau_0\bar{\Xi}_j' - b(\tau_0)\int_{c_k}^{\tau_0} 2i\,\mathrm{Im}\,\Xi(\tau, \bar{\tau}; t_j, \bar{t}_j)\frac{d\tau + d\bar{\tau}}{\tau - \bar{\tau}}$$

$$\Theta_j = \Theta(\tau_0, \bar{\tau}_0; t_j, \bar{t}_j), \qquad \Xi_j = \Xi(\tau_0, \bar{\tau}_0; t_j, \bar{t}_j), \qquad A = 0$$

and when $\tau_0 \in l_k' \in \Lambda_1$

$$a(\tau_0)=\frac{1}{4}(3-2\varkappa'),\qquad b(\tau_0)=\frac{1}{4}(1+2\varkappa'),\qquad f(\tau_0)=R_k+\frac{2}{\tau_0-\bar\tau_0}Z_k$$
$$C(\tau_0)=C_k+\frac{2(1-\nu)}{\tau_0-\bar\tau_0}C_k'-2(1-\nu)C_k'\int\limits_{c_k}^{\tau_0}\frac{d\tau+d\bar\tau}{(\tau-\bar\tau)^2} \tag{2.3}$$

whereas, for $\tau_0\in l_k'\in\Lambda_2$

$$\alpha(\tau_0)=\frac{1}{2}-\varkappa',\qquad b(\tau_0)=0,\qquad f(\tau_0)=2G(w+iu),\qquad C(\tau_0)=0. \tag{2.4}$$

Z_k and R_k may be obtained from (1.6), where c_k is the initial point in case x_j' is not an end point of l_k'; if the opposite is true, x_j' is taken as the initial point in integrating (1.6) (in the negative direction). In general, the number of real constants C_k and C_k' equals p; $C_k'=0$ for curve segments adjoining the z-axis. $\alpha_j(\tau_0)$ equals unity if $c_k=x_j$ $(j=m+1,\ldots n)$, and zero in all other cases. Under these conditions, the left- and right-hand sides of (2.1) are continuous over the interval of each curve l_k' (the branch cuts for Θ_j and Ξ_j $(j=1,2,\ldots,m)$ pass through one of the points $c_k\in L_j'$).

Following the ideas of Sherman [9], we will represent the regular part of (1.7) in the form of generalized Cauchy type integrals

$$\Phi_*(t,\bar t)=\frac{1}{2\pi i}\int\limits_L F(\tau)\,W d\tau\qquad\left(W=W(t,\tau)=\frac{\psi(t,\tau)}{\tau-t}\right)$$
$$\Psi_*(t,\bar t)=\frac{1}{4\pi i}\int\limits_L\overline{F(\tau)}\,W\,[(1-2\varkappa')\,d\tau+d\bar\tau]-\frac{1}{2\pi i}\int\limits_L F(\tau)\left(\bar\tau\frac{\partial W}{\partial z}d\tau-W d\bar\tau\right) \tag{2.5}$$

where the weight function $F(\tau)$ satisfies (1.3), W is the generalized Cauchy kernal [6], $\psi(t,\tau)$ is defined by the equations

$$\psi(t,\tau)=\left|\frac{\tau-\bar\tau}{\tau-\bar t}\right|[K(k_1)-D(k_1)],\qquad \operatorname{Im}\tau\cdot\operatorname{Im}t\geqslant 0$$
$$\psi(t,\tau)=\left|\frac{\tau-\bar\tau}{\tau-t}\right|D(k_2),\qquad \operatorname{Im}\tau\cdot\operatorname{Im}t\leqslant 0 \tag{2.6}$$
$$D(k)=\frac{1}{k^2}[K(k)-E(k)],\quad k_1=\frac{\sqrt{|\tau-\bar\tau|\cdot|t-\bar t|}}{|\tau-\bar t|},\quad k_2=\frac{\sqrt{|\tau-\bar\tau|\cdot|t-\bar t|}}{|\tau-t|}$$

Then

$$-\varkappa'\Phi_*(t,\bar t)+t\overline{\Phi_*'(t,\bar t)}+\overline{\Psi_*(t,\bar t)}=\frac{1-2\varkappa'}{4\pi i}\int\limits_L F(\tau)(W d\tau-\overline{W}d\tau)+$$
$$+\frac{1}{2\pi i}\int\limits_L\overline{F(\tau)}\left[(\tau-t)\frac{\partial\overline{W}}{\partial z}d\bar\tau-\overline{W}d\tau\right]-\frac{1}{4\pi i}\int\limits_L F(\tau)(W+\overline{W})\,d\tau=$$
$$=\frac{1-2\varkappa'}{4\pi i}\int\limits_L F(\tau)(W d\tau-\overline{W}d\bar\tau)-\frac{1}{2\pi i}\int\limits_L F(\tau)\,\overline{W}_1\left(\frac{\bar\tau-t}{\tau-\bar t}d\tau-d\bar\tau\right)- \tag{2.7}$$
$$-\frac{1}{4\pi i}\int\limits_L F(\tau)\frac{\tau+\bar\tau-t-\bar t}{\tau-\bar t}(W+\overline{W})\,d\tau$$

Here, $W_1=W(t,\bar\tau)$, and we have taken into account

$$\frac{\partial W}{\partial z} = \frac{1}{2(\tau - t)}(2W + W_1 + \overline{W}_1), \qquad \int_L F(\bar\tau) W_1 d\bar\tau = -\int_L F(\tau) W d\tau$$

Noting that the Sokhotskii-Plemelj formulas hold for generalized Cauchy type integrals, we let t approach τ_0 in (2.5) and (2.7). Substituting the resultant expressions in (2.1), we obtain

$$A_j = \operatorname{Re}\int_{L'_j} F(\tau)\, ds = \frac{1}{2}\int_L F(\tau)\rho_j(\tau)\, d\tau$$

$$B_j = \operatorname{Im}\int_{L'_j} F(\tau)\,|\tau - \bar\tau|\, ds = -\frac{1}{2}\int_L F(\tau)\,(\tau - \bar\tau)\rho_j(\tau)\, d\tau \tag{2.8}$$

Here $\rho_j(\tau) = d\bar\tau/ds$ for $\tau \in L_j$, and $\rho_j = 0$ in all other cases. As a result, (2.1) becomes a singular integral equation in $F(\tau)$:

$$a(\tau_0)F(\tau_0) + \frac{b(\tau_0)}{\pi i}\int_L W(\tau_0, \tau)F(\tau)\,d\tau + \int_L K(\tau_0, \tau)F(\tau)\,d\tau = -f(\tau_0) + C(\tau_0) \tag{2.9}$$

$$K(\tau_0, \tau) = \frac{2\varkappa' - 1}{4\pi i}\psi(\tau_0, \tau)\frac{d}{d\tau}\ln\frac{\bar\tau - \bar\tau_0}{\tau - \tau_0} + \frac{1}{2\pi i}\psi(\tau_0, \bar\tau)\frac{d}{d\tau}\left(\frac{\bar\tau - \tau_0}{\tau - \bar\tau_0}\right) -$$

$$-\frac{1}{4\pi i}\psi(\tau_0, \tau)\frac{(\tau + \bar\tau - \tau_0 - \bar\tau_0)^2}{(\tau - \bar\tau_0)\,|\tau - \tau_0|^2} - \frac{1}{2(\tau - \bar\tau)}b(\tau_0)\left(1 + \frac{d\bar\tau}{d\tau}\right)[\beta(\tau_0, \tau) - \beta(\tau_0, \bar\tau)] -$$

$$-\frac{b(\tau_0)}{\pi i}Q(\tau_0, \tau) + \frac{1}{2}\sum_{j=1}^{n} S_j(\tau_0)\rho_j(\tau) - \frac{1}{2}\sum_{j=1}^{m} T_j(\tau_0)(\tau - \bar\tau)\rho_j(\tau) \tag{2.10}$$

$$Q(\tau_0, \tau) = \int_{c_k}^{\tau_0} [W(t, \tau) - \overline{W(t, \bar\tau)}]\,\frac{dt + d\bar t}{t - \bar t} \qquad (t, \tau_0 \in l_k')$$

Here $\beta(\tau_0, \tau) = 1$ if $\tau \in c_k\tau_0$ with $\tau_0 \in l_k'$, and $\beta(\tau_0, \tau) = 0$ for other relative arrangements between τ_0 and τ. In differentiating with respect to τ, it should be kept in mind that $\bar\tau$ is a function of τ while τ_0 and $\bar\tau_0$ are constants. Although it was assumed above that $\operatorname{Im}\tau_0 \geqslant 0$, it turns out that (2.9) holds for arbitrary τ_0, provided that

$$a(\tau_0) = a(\bar\tau_0),\ b(\tau_0) = b(\bar\tau_0),\ f(\tau_0) = \overline{f(\bar\tau_0)},\ C(\tau_0) = \overline{C(\bar\tau_0)},\ K(\tau_0, \tau) = -\overline{K(\bar\tau_0, \bar\tau)} \tag{2.11}$$

The kernel $K(\tau_0, \tau)$ is given by

$$K(\tau_0, \tau) = \frac{N(\tau_0, \tau)}{|\tau - \tau_0|^\lambda\,|\tau - \bar\tau_0|^\lambda}\prod_{(k)}(|\tau - c_k|\cdot|\tau - \bar c_k|)^{-\lambda} \tag{2.12}$$

where the continued product symbol extends over all nodes, $N(\tau_0, \tau)$ is a function of class H, and λ is a number in the range $0 < \lambda < 1$, which can be made arbitrarily small.

Note that Equation (2.9) may be transformed into an integral equation with the usual Cauchy kernel

$$a(\tau_0)F(\tau_0) + \frac{b(\tau_0)}{\pi i}\int_L \frac{F(\tau)\,d\tau}{\tau - \tau_0} + \int_L\left[\frac{1}{\pi i}K_0(\tau_0, \tau) + K(\tau_0, \tau)\right]F(\tau)\,d\tau = -f(\tau_0) + C(\tau_0) \tag{2.13}$$

The kernel $K_0(\tau_0, \tau) = b(\tau_0)[\psi(\tau_0, \tau) - 1](\tau - \tau_0)^{-1}$ does not contribute any new singularities to (2.12) if none of the segments $l_k' \in \Lambda_1$ adjoin the z-axis. However, if, for example, $x_j \in \Lambda_1$, then, for $\tau_0 = \bar{\tau}_0 = x_j$ and $\tau \to x_j$, this kernel has a singularity of the type const $(\tau - x_j)^{-1}$.

3. Suppose $f(\tau)$ is of class H_0, and $df/d\tau$ is of class H^*. We seek a solution $F(\tau)$ to (2.9), belonging to class h_{2p} (the terminology is that of [11]). By using the method of [10] together with certain additional considerations, it can be shown that $F(\tau)$ is of H, and the derivative $dF/d\tau$ is of class H^*. It follows from this that the expressions in the right-hand sides of (1.1) and (1.5) extend continuously to all points of every segment l_k' and the computations of section 2 are valid.

Consider Equation (2.9) for $f(\tau_0) \equiv 0$. Let $F_0(\tau)$ be the solution of this equation, satisfying (1.3), and let $\Phi_0(t, \bar{t})$ and $\Psi_0(t, \bar{t})$ be the corresponding generalized analytic functions. It can be shown that the stresses obtained by substituting Φ_0 and Ψ_0 into (1.4) may be continuously extended to all points on the boundary L, except for the nodes. In the neighborhood of each node c_k, their magnitudes do not exceed const $|t - c_k|^{-\alpha} (\alpha < 1)$. Thus, the uniqueness theorem holds. Consequently

$$\Phi_0(t, \bar{t}) = \gamma + \frac{1}{t - \bar{t}}\gamma', \quad \Psi_0(t, \bar{t}) = \varkappa'\gamma - \frac{\varkappa'}{t - \bar{t}}\gamma', \quad C_k \equiv \gamma, \quad C_k' \equiv \gamma' \quad (3.1)$$

Here γ and γ' are real constants. Substituting the expressions obtained for Φ and Ψ into (1.7) while taking into acount that $A = 0$ and that Φ_* and Ψ_* are regular, we obtain $\gamma' = 0$ and

$$A_j \equiv 0, \qquad B_j \equiv 0. \quad (3.2)$$

Utilization of (2.5), the second equation in (2.5) being modified via integration by parts, yields

$$\gamma = \frac{1}{2\pi i}\int_L F_0(\tau) W d\tau \quad (3.3)$$

$$\varkappa'\gamma = \frac{1}{2\pi i}\int_L \left\{(1 - \varkappa')\overline{F_0(\tau)} - \bar{\tau}\frac{d}{d\tau}F_0(\tau) - \frac{\bar{\tau}}{2(\tau - \bar{\tau})}\left(1 - \frac{d\bar{\tau}}{d\tau}\right)[F_0(\tau) - \overline{F_0(\tau)}]\right\} W d\tau$$

If we now introduce the notation

$$\Phi^*(\tau, \bar{\tau}) = F_0(\tau) - \gamma$$
$$\Psi^*(\tau, \bar{\tau}) = (1 - \varkappa')\overline{F_0(\tau)} - \bar{\tau}\frac{d}{d\tau}F_0(\tau) - \frac{\bar{\tau}}{2(\tau - \bar{\tau})}\left(1 - \frac{d\bar{\tau}}{d\tau}\right)[F_0(\tau) - \overline{F_0(\tau)}] - \varkappa'\gamma \quad (3.4)$$

it follows from (3.3) that $\Phi^*(\tau, \bar{\tau})$ and $\Psi^*(\tau, \bar{\tau})$ are the boundary values of $\Phi^*(t, \bar{t})$ and $\Psi^*(t, \bar{t})$, which are regular in the regions $D_1', D_1'', D_2', D_2'', \ldots, D_m', D_m'', D_{m+1}', \ldots, D_n, D_{n+1}$ and vanish at infinity (the finite regions D_j' and D_j'' $j = 1, 2, \ldots, m$ lie within the respective L_j' and L_j''; the regions D_j $[j = m+1, \ldots, n]$ are within the contours L_j, and the infinite region D_{n+1} lies outside the contour L_{n+1}).

Eliminating $F_0(\tau)$ from (3.4), we obtain

$$(1 - \varkappa')\overline{\Phi^*(\tau, \bar{\tau})} - \bar{\tau}\,\Phi^{*\prime}(\tau, \bar{\tau}) - \Psi^*(\tau, \bar{\tau}) = 8(1 - \nu)\gamma \quad (3.5)$$

Here, $\Phi^{*\prime}(\tau, \bar{\tau})$ represents the boundary value of $\Phi^{*\prime}(t, \bar{t})$. Application of Green's theorem readily yields

$$\operatorname{Im}\int_{L'_j} \Phi^{*\prime}[(1 - \varkappa')\overline{\Phi^*} - \bar{\tau}\Phi^{*\prime} - \Psi^*]\,|\tau - \bar{\tau}|\,d\tau = -8\iint_{D'_j}[2(1 - \nu)(\operatorname{Re}\Phi^{*\prime})^2 +$$
$$+ (1 - 2\nu)(\operatorname{Im}\Phi^{*\prime})^2]\,r\,dz\,dr \qquad (j = 1, 2, \ldots, n+1) \quad (3.6)$$

With the aid of (3.5), the left-hand side of (3.6) can be seen to vanish. Thus, $\Phi^{*\prime}(t, \bar{t}) \equiv 0$, and

$$\Phi^*(t, \bar{t}) = \gamma_j + \frac{1}{t-\bar{t}}\gamma_j', \quad \Psi^*(t, \bar{t}) = -8(1-\nu)\gamma + (1-\varkappa')\gamma_j - \frac{1-\varkappa'}{t-\bar{t}}\gamma_j', \; t \in D_j' \tag{3.7}$$

Here γ_j and γ_j' are real constants, and $\gamma_j' = 0$ $(j \geqslant m+1)$.

From the conditions at infinity, it follows that

$$\gamma_{n+1} = 0, \qquad \gamma = 0 \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.4), we obtain

$$F_0(\tau) = \gamma_j + \frac{1}{\tau - \bar{\tau}}\gamma_j' \qquad (\tau \in L_j' + L_j''; \; j = 1, 2, \ldots m)$$

$$F_0(\tau) = \gamma_j \qquad (\tau \in L_j; \; j = m+1, \ldots n), \qquad F_0(\tau) = 0 \qquad (\tau \in L_{n+1}) \tag{3.9}$$

Now, utilization of (2.8) and (3.2) yields $\gamma_j \equiv 0$ and $\gamma_j' \equiv 0$, whence

$$F_0(\tau) \equiv 0, \qquad C_k \equiv 0, \qquad C_k' \equiv 0 \tag{3.10}$$

In a similar manner it may be shown that the homogeneous equation (2.9) has no solution other than the trivial one. Condition (1.3) on $F(\tau)$ does not restrict generality, since, as a result of (2.11), the arbitrary solution of (2.9) may be written in the form $F_1(\tau) + iF_2(\tau)$, where $F_1(\tau)$ and $F_2(\tau)$ satisfy (2.9) and (1.3).

The remaining discussion of this section is based on the assumption that Noether's theorems hold for Equation (2.9) [or (2.13)] (if no point on the axis of symmetry belongs to Λ_1, then this is obvious, for in that case K_0 is a Fredholm kernel).

The index of the class h_{2p} of Equation (2.9) equals $(-p)$. Hence, there exist p linearly independent solutions of class h_0 to the associated homogeneous equation

$$a(\tau_0)\,\mathrm{K}(\tau_0) + \frac{1}{\pi i}\int_L b(\tau)\,W(\tau, \tau_0)\,\mathrm{K}(\tau)\,d\tau + \int_L K(\tau, \tau_0)\,\mathrm{K}(\tau)\,d\tau = 0 \tag{3.11}$$

Without restricting generality, these solutions $K_j(\tau)$ $(j = 1, 2, \ldots, p)$ may be taken to satisfy (1.3).

Now the conditions for the existence of a solution to (2.9) take the form

$$\int_L [C(\tau) - f(\tau)]\,\mathrm{K}_j(\tau)\,d\tau = 2i\,\mathrm{Im}\int_{L'} [C(\tau) - f(\tau)]\,\mathrm{K}_j(\tau)\,d\tau = 0 \quad (j = 1, 2, \ldots p) \tag{3.12}$$

Hence, one may easily obtain a system of real, linear equations for the determination of the constants C_k and C_k'. Proceeding in a manner similar to [10] and taking into account (3.1) to (3.10), we can show that the determinant of this system is nonzero. Consequently the posed problem has a real solution.

4. Let us remove the restrictions placed on Λ_1 in section 2. Let for example, the contour L_m' be entirely in Λ_1. Then, for $j = m$, we write in place of (2.8) (cf. [7]):

$$A_m = \frac{1}{4\pi(1-\nu)}\int_{L'_m} rp_z ds, \quad 4\pi B_m + \int_{L'_m}\left[2i\,\mathrm{Im}\,\Phi(\tau, \bar{\tau}) - \frac{1}{\tau - \bar{\tau}}C_m'\right]\frac{d\tau + d\bar{\tau}}{\tau - \bar{\tau}} =$$
$$= \frac{1}{2(1-\nu)}\int_{L'_m}\left(p_r + \frac{1}{r^2}Z_m\frac{dz}{ds}\right)ds \tag{4.1}$$

$$C_m = -\operatorname{Re}\int_{L'_m} F(\tau)\,ds, \qquad C_m' = -\operatorname{Im}\int_{L'_m} F(\tau)\,|\tau-\bar{\tau}|\,ds$$

Both equations in (2.5) must be supplemented by terms

$$b_m\Theta'(t, \bar{t};\ t_m, \bar{t}_m) \qquad \left(b_m = \operatorname{Re}\int_{L'_m} \overline{F(\tau)}(\tau-\bar{\tau})\,d\tau\right) \tag{4.2}$$

and $\varkappa'$ therein must be replaced by $\varkappa_1(\tau) = -0.5$ for $\tau \in L_m' + L_m''$ while $\varkappa_1(\tau) = \varkappa'$ for the remaining parts of L. The resultant equation is of the form (2.9), with $b(\tau_0) = 0$ for $\tau_0 \in L_m' + L_m''$. The existence of a solution is then readily shown in the same manner as in section 3.

The indicated method may be extended to a case with several contours $L_j' \in \Lambda_1$ (some of which may adjoin the axis of symmetry).

BIBLIOGRAPHY

1. Rostovtsev N.A., Kompleksnye funktsii napriazhenii v osesimmetrichnoi kontaktnoi zadache teorii uprugosti (Complex stress functions in the axisymmetric contact problem of elasticity). *PMM*, Vol. 17, No. 5, 1953.

2. Aleksandrov, A.Ia. and Solov'ev, Iu.I., Odna forma resheiia prostranstvennykh osesimmetrichnykh zadach teorii uprugosti pri pomoshchi funktsii kompleksnogo peremennogo i resheniie etikh zadach dlia sfery (A form of the solution of three-dimensional axisymmetric problems of elasticity in terms of complex variables and the solution of these problems for a sphere). *PMM* Vol. 26, No. 1, 1962.

3. Aleksandrov A.Ia. and Solov'ev Iu.I., Odna forma resheiia prostranstvennykh osesimmetrichnykh zadach teorii uprugosti pri pomoshchi funktsii kompleksnogo peremennogo i resheniie etikh zadach dlia sfery (A form of the solution of three-dimensional axisymmetric problems of elasticity in terms of complex variables and the solution of these problems for a sphere). *PMM*. Vol. 26, No. 1, 1962.

4. Polozhii G.N., O (*p*, *q*) – analiticheskikh funktsiiakh kompleksnogo peremennogo i nekotorykh ikh primeneniiakh (On (*p*, *q*) – analytic functions of a complex variable and some of their applications). In 'Issledovaniia po sovremennym problemam teorii funktsii kompleksnogo peremennogo' Fizmatgiz, 1960.

5. Chemeris V.S., Ob integral'nykh uravneniiakh osesimmetrichnoi theorii uprugosti (On integral equations in axisymmetric theory of elasticity). Prikladnaia mekhanika, Akad. Nauk USSR, Vol. 1, No. 5, 1965.

6. Solov'ev Iu.I., Reshenie osesimmetrichnoi zadachi teorii uprugosti dlia odnosviaznykh tel vrashcheniia (Solutions of axisymmetric problems in elasticity for simply connected bodies of revolution). Inzh. zh. Vol. 5, No. 3, 1965.

7. Solov'ev Iu.I., Reshenie prostransvennoi osesimmetrichnoi zadachi teorii uprugosti dlia mnogosviaznykh tel vrashcheniia pri pomoshchi obobshchennykh analiticheskikh funktsii (Solution of three-dimensional axisymmetric problems in elasticity for multiply connected bodies of revolution, using generalized analytic functions). Dokl. Akad. Nauk SSSR, Vol. 169, No. 2, 1966.

8. Daniliuk I.I., Ob obshchem predstavlenii osesimmetricheskikh polei (On a general representation of axisymmetric fields). PMTF, No. 2, 1960.

9. Sherman D.A., Smeshannaia zadacha staticheskoi teorii uprugosti dlia ploskikh mnogosviaznykh oblastei (The mixed problem of static elasticity for plane multiply connected regions). Dokl. Akad. Nauk SSSR, Vol. 28, No. 1, 1940.

10. Mandzhavidze G.F., Ob odnom singuliarnom integral'nom uravnenii s razryvnymi coeffitsientami i ego primenenii v teorii uprugosti (On a singular integral equation with discontinuous coefficients and its application in elasticity). *PMM*. Vol. 15, No. 3, 1951.

11. Muskhelishvili N.I., Singuliarnye integral'nye uravnenia (Singular Integral Equations). Izd. 2-e, M., Fizmatgiz, 1962.

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